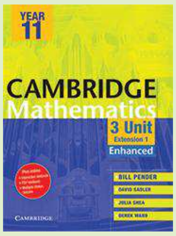
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1) YEAR 12 COURSE

Topic summary and exercises:

With references to



(A) (X1) **Sequences and series**

Name: ....

Initial version by H. Lam, September 2014. Last updated December 4, 2020 for latest syllabus.
Various corrections by students & members of the Department of Mathematics at North Sydney Boys and Normanhurst Boys High Schools.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under

Symbols used

- (!) Beware! Heed warning.
- (F) Provided on NESA Reference Sheet
- (M) Facts/formulae to memorise.
- (2) Textbook reference from the legacy Mathematics (2 Unit) course
- (A) Mathematics Advanced content.
- (X1) Mathematics Extension 1 content.
- (L) Literacy: note new word/phrase.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA12-4 applies the concepts and techniques of arithmetic and geometric sequences and series in the solution of problems

Syllabus subtopics

MA-M1 (1.2, 1.3) Modelling Financial Situations (Arithmetic sequences and series, Geometric sequences and series)

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *Cambridge Year 11 3 Unit* (Pender, Sadler, Shea, & Ward, 1999) or *Cambridge Year 11 2 Unit* (Pender, Sadler, Shea, & Ward, 2009) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Part I

Terms of sequences

Section 1

General sequences

1.1 Notation

Important note

The n-th term of a sequence is denoted T_n.

Example 1

Find:

(a) the first five terms (b) the formula for the n-th term

of the sequence defined by

$$T_1 = 1 \qquad T_n = \frac{n-1}{n} T_{n-1}$$

for $n \geq 2$.

Answer: $T_n = \frac{1}{n}$

Example 2

[Ex 6C Q10]

(a) Find whether $1\frac{1}{2}$ and 96 are members of the sequence $T_n = \frac{3}{32} \times 2^n$, and if so, what terms they are.

(b) Find the first term in this sequence which is greater than 10.

Example 3

[Ex 6C Q13] A sequence satisfies $T_n = \dfrac{1}{2}\left(T_{n-1} + T_{n+1}\right)$ with $T_1 = 3$ and $T_2 = 7$. Find T_3 and T_4.

Further exercises

(2) **Ex 8A** (Pender et al., 2009)

- Q1-2 last column
- Q3-5 (e) onwards
- Q6-11

(X1) **Ex 6C** (Pender et al., 1999)

- Q2-15

Section 2

Terms of arithmetic sequences

Definition 1

An *arithmetic sequence* (or *arithmetic progression*, AP) is defined by

$$T_n - T_{n-1} = d$$

for $n \geq 2$, where d is a constant, known as the *common difference*.

Important note

T_1 is given the symbol a.

2.1 Derivation of T_n formula

- $T_1 = a$
- $T_2 = T_1 + d = a + d$
- $T_3 = T_2 + d = a + 2d$
- $T_4 = T_3 + d = a + 3d$

$$\vdots$$

Laws/Results

Ⓕ The n-th term of an arithmetic sequence is

$$T_n = a + (n-1)d$$

2.2 Proof of arithmetic sequence

Important note

Condition for AP: $T_3 - T_2 = T_2 - T_1$

Example 4

Show that the sequence $200, 193, 186, \ldots$ is an AP. Then find a formula for the n-th term, and find the first negative term. **Answer:** $T_{30} = -3$

Example 5

Show that $\log_5 6$, $\log_5 12$, $\log_5 24$ is an AP.

2.3 Other examples

Example 6

The third term of an AP is 16, and the 12th term is 79. Find the 41st term.

Answer: 282

Further exercises

(2) **Ex 8B** (Pender et al., 2009)

- Q2-5, 11-12 last 2 columns
- Q6-10, 13-19

(x1) **Ex 6D** Pender et al. (1999)

- Q1-14, last column

Section 3

Terms of geometric sequences

Definition 2

A *geometric sequence* (or *geometric progression*, GP) is defined by

$$\frac{T_n}{T_{n-1}} = r$$

for $n \geq 2$, where r is a constant, known as the *common ratio*.

3.1 Derivation of T_n formula

Steps

1. $T_1 = a$
2. $T_2 = T_1 \times r = ar$
3. $T_3 = T_2 \times r = ar^2$
4. $T_4 = T_3 \times r = ar^3$

$\vdots$

Laws/Results

(F) The n-th term of a geometric sequence is

$$T_n = ar^{n-1}$$

3.2 Proof a geometric sequence

Important note

Condition for GP: $\dfrac{T_3}{T_2} = \dfrac{T_2}{T_1}$

Example 7

Find the value(s) of x such that 3, $x + 4$ and $x + 10$ form:

(a) arithmetic

(b) geometric

sequence.

Answer: (a) $x = 5$ (b) $x = 2$ or -7

3.3 Other examples

Example 8

Find the first term a and the common ratio r of a GP where the fourth term is 30 and the sixth term is 480.

Answer: $r = \pm 4$, $a = \pm \frac{15}{32}$

Example 9

(a) Show that the sequence whose terms are 1 000, 400, 160... form a GP, and then find the formula for the n-th term.

(b) Find the first term less than $\frac{1}{1\,000}$.

Answer: (a) Show (b) T_{17}

Example 10

[**2008 2U**] The zoom function in a software package multiplies the dimensions of an image by 1.2. In an image, the height of a building is 50 mm. After the zoom function is applied once, the height of the building in the image is 60 mm. After a second application, its height is 72 mm.

(i) Calculate the height of the building in the image after the zoom function has been applied eight times. Give your answer to the nearest mm. **2**

(ii) The height of the building in the image is required to be more than 400 mm. Starting from the original image, what is the least number of times the zoom function must be applied? **2**

Further exercises

(2) **Ex 8C/D** (Pender et al., 2009)

- Ex 8C: Q2-5, 9-11 last 2 columns, Q6-8, 12-17
- Ex 8D: Q1-3, 15 last 2 columns, Q4-14

(x1) **Ex 6E** (Pender et al., 1999)

- Q1-20, last column

Part II

Series: sums of sequences

Section 4

Sums of general sequences

4.1 Sigma notation

Definition 3

Sum of terms from ordinal k to ordinal ℓ.

$$\sum_{n=k}^{n=\ell} T_n = T_k + T_{k+1} + T_{k+2} + \cdots + T_{\ell-2} + T_{\ell-1} + T_\ell$$

Example 11

Evaluate $\sum_{k=1}^{5} k$.

Example 12

[Ex 6G Q1(b)] Evaluate $\sum_{n=1}^{5} n^2$

Example 13

Evaluate $\sum_{n=4}^{7}(5n+1)$ **Answer:** 114

Example 14

Evaluate $\sum_{n=1}^{5} 3 \times (-2)^n$ **Answer:** -66

Example 15

Express $\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12}$ in sigma notation.

Further exercises

Ex 6G (Pender et al., 1999)

- Q1 last 3 columns
- Q2-3

4.2 Partial sums

Definition 4

The n-th partial sum of a sequence

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n$$

Important note

(M) To recover the n-th term from a sum,

$$S_n = \underline{S_{n-1}} + \underline{T_n}$$

Rearrange to find T_n

Example 16

Given $S_n = n^2$, find a formula for the n-th term.

Further exercises

(2) **Ex 8E** (Pender et al., 2009)

- Q1-4 last column
- Q8-10

(x1) **Ex 6H** (Pender et al., 1999)

- Q3, 4, 7
- Q8 last 2 columns

Section 5

Arithmetic Series

5.1 Derivation of S_n formula

Steps

1. Sum to n terms:

$$S_n = T_1 + T_2 + \cdots + \overbrace{T_n}^{\ell}$$

$$= a + (a+d) + \cdots + (\ell - d) + \ell$$

2. Reverse the sum:

$$S_n = T_n + \cdots + T_2 + T_1$$

$$= a + (a+d) + \cdots + (\ell - d) + \ell$$

3. Add the two sums:

4. Given ℓ is also T_n:

Laws/Results

Ⓕ The sum of an arithmetic progression to n terms:

$$S_n = \frac{n}{2}(a+\ell)$$

$$= \frac{n}{2}\left(2a+(n-1)d\right)$$

5.2 Examples

Example 17

Add up all the integers from 100 to 200 inclusive. **Answer:** 15 150

Example 18

(a) Given the AP $40 + 37 + 34 + \cdots$, find S_{10}.

(b) What is the first negative partial sum?

Answer: $S_{28} = -14$.

Example 19

The sum of the first ten terms of an AP is zero, and the sum of the first and second terms is 24. Find the first three terms. **Answer:** $\frac{27}{2}$, $\frac{21}{2}$, $\frac{15}{2}$

Example 20

[Ex 6I Q8(f)] Find the sum $\sqrt{12}+\sqrt{27}+\sqrt{48}+\cdots+21\sqrt{3}$.

Answer: $230\sqrt{3}$

Example 21

[Ex 6I Q12(c)] Find the sum of $\log_b 36+\log_b 18+\log_b 9+\cdots\log_b \frac{9}{8}$.

Answer: $3\left(4\log_b 3-\log_b 2\right)$

Example 22

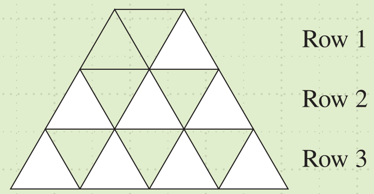
The first term of an arithmetic sequence is 5. The ratio of the sum of the first four terms to the sum of the first ten terms is 8 : 35. Find the common difference.

Answer: $d=2$

Example 23

[**2012 2U**] Jay is making a pattern using triangular tiles. The pattern has 3 tiles in the first row, 5 tiles in the second row, and each successive row has 2 more tiles than the previous row.

Answer: 13 complete rows



(i) How many tiles would Jay use in row 20? **2**

(ii) How many tiles would Jay use altogether to make the first 20 rows? **1**

(iii) Jay has only 200 tiles. **2**

How many complete rows of the pattern can Jay make?

Example 24

[2000 2U] (6 marks) In the construction of a 5 km expressway a truck delivers materials from a base. After depositing each load, the truck returns to the base to collect the next load. The first load is deposited 200 m from the base, the second 350 m from the base, the third 500 m from the base. Each subsequent load is deposited 150 m from the previous one. **Answer:** (i) 2 300 m (ii) 33 (iii) 171.6 km

(i) How far is the fifteenth load deposited from the base?

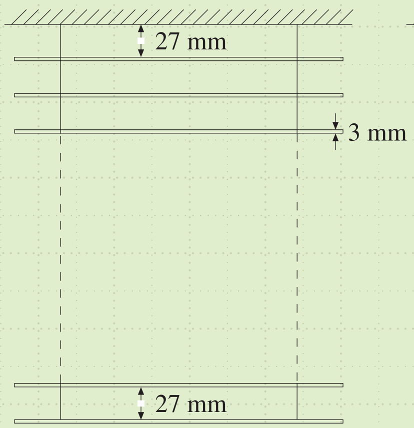
(ii) How many loads are deposited along the total length of the 5 km expressway? (The last load is deposited at the end of the expressway.)

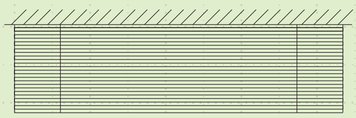
(iii) How many kilometres has the truck travelled in order to make all the deposits and then return to the base?

Example 25

[1996 2U] (5 marks) A venetian blind consists of twenty-five slats, each 3 mm thick. When the blind is down, the gap between the top slat and the top of the blind is 27 mm and the gap between adjacent slats is also 27 mm, as shown in the first diagram.

Answer: Total: 8 775 mm





When the blind is up, all the slats are stacked at the top with no gaps, as shown in the second diagram.

(i) Show that when the blind is raised, the bottom slat rises 675 mm.

(ii) How far does the next slat rise?

(iii) Explain briefly why the distances the slats rise form an arithmetic sequence.

(iv) Find the sum of all the distances that the slats rise when the blind is raised.

Example 26

[**2014 Independent Q14**] Felicity receives a money box on the day she's born.

Her parents decide that each month, on the 1st of the month, they will deposit money into her money box and give her this money on her 21st birthday.

The first month they deposit \$10 into the money box.
The second month they deposit \$20, the third month they deposit \$30.
Each month they deposit \$10 more into the money box than they did the month before.

Answer: (i) 252 (ii) \$2 520 (iii) \$318 780

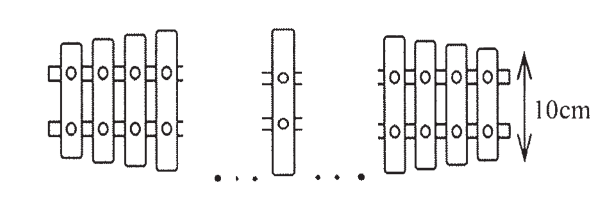
(i) How many times over the 21 years will Felicity's parents deposit money into her money box? **1**

(ii) How much will be deposited into the money box in the month of her 21st birthday? **2**

(iii) How much will Felicity receive from her parents on her 21st birthday? **2**

Example 27

[2014 CSSA 2U] A percussionist is experimenting with designs for a xylophone. It is to be symmetrical in shape as shown in the diagram.



The shortest wooden bar is to be 10 cm long and the consecutive bars will differ in length by b cm. The total length of all the wooden bars is S cm. Let the number of wooden bars be $2n + 1$. **Answer:** 21

(i) Show that $S = bn^2 + 20n + 10$ **2**

(ii) Given that $S = 360$ cm and $b = 1.5$ cm, find the number of wooden bars. **2**

Further exercises

(2) **Ex 8F** (Pender et al., 2009)

- Q1-8 last column
- Q9, 11-16

(x1) **Ex 6I** (Pender et al., 1999)

- Q2-5 last column
- Q6-16 even #

Section 6

Geometric Series

6.1 Derivation of S_n formula

Steps

1. Sum to n terms:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \tag{6.1}$$

2. Multiply by r:

$$rS_n = a + ar^2 + \cdots + ar^{n-1} \tag{6.2}$$

3. Subtract (6.1) from (6.2):

4. Factorise, and change subject to S_n:

Laws/Results

(F) The sum of a geometric progression to n terms:

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1} = \frac{a\left(1 - r^n\right)}{1 - r}$$

6.2 Examples

Example 28

Find the sixth partial sum of the GP $2 - 6 + 18 - \cdots$ **Answer:** -364

Example 29

How many terms of the GP $2 + 6 + 18 + \cdots$ must be taken for the sum to exceed one billion? **Answer:** 19

Example 30

[2016 2U] (2 marks) By summing the geometric series $1 + x + x^2 + x^3 + x^4$ or otherwise, find $\lim_{x \to 1} \dfrac{x^5 - 1}{x - 1}$.

(!) Good setting out is crucial!

Example 31

[2011 2U] The number of members of a new social networking site doubles every day. On Day 1 there were 27 members and on Day 2 there were 54 members.

(i) How many members were there on Day 12? **1**

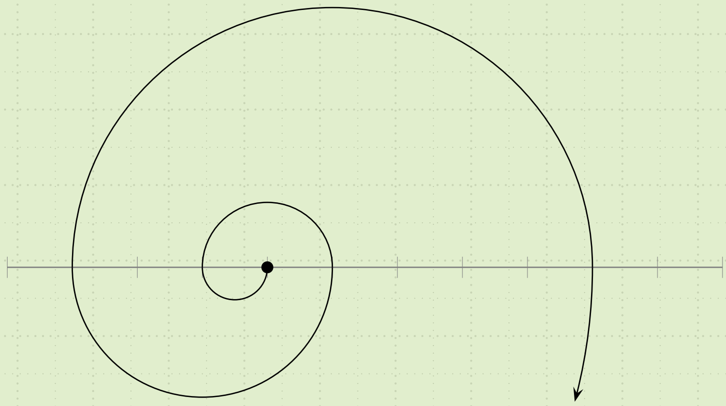
(ii) On which day was the number of members first greater than 10 million? **2**

(iii) The site earns 0.5 cents per member per day. How much money did the site earn in the first 12 days? Give your answer to the nearest dollar. **2**

Example 32

[**2011 NEAP 2U**] The spiral below is formed by connecting semicircles of increasing radii.

Answer: (i) Show (ii) $1\,984\pi$



(i) Show that the length of this spiral generates the geometric series: **2**

$$\pi \sum_{k=1}^{n} 2^{k-2} = \frac{\pi\left(2^n - 1\right)}{2}$$

if the length of the first and second semicircles are $\dfrac{\pi}{2}$ and π respectively.

(ii) Calculate the length of the spiral between the start of the 8th and the end of the 12th semicircles. **2**

Example 33

[**2013 2U**] (!) Kim and Alex start jobs at the beginning of the same year. Kim's annual salary in the first year is \$30 000, and increases by 5% at the beginning of each subsequent year. Alex's annual salary in the first year is \$33 000, and increases by \$1 500 at the beginning of each subsequent year. **Answer:** (i) Show (ii) \$377 336.78 (iii) $n = 7$

(i) Show that in the 10th year Kim's annual salary is higher than Alex's annual salary. **2**

(ii) In the first 10 years how much, in total, does Kim earn? **2**

(iii) Every year, Alex saves $\frac{1}{3}$ of her annual salary. How many years does it take her to save \$87 500? **3**

Further exercises

(2) **Ex 8G** (Pender et al., 2009)

- Q3-5 last column
- Q7-8, 10-12

(x1) **Ex 6J** (Pender et al., 1999)

- Q3-4 last column
- Q5
- Q6-7 last column
- Q9-17 odd #

Section 7

Limiting sum

7.1 Existence of a limiting sum

Example 34

A frog is about to jump over a river that is 10 m wide. The first jump it can make is 5 m (where there will be an object to help it stay above water), and each subsequent jump it can only jump half as much as the previous jump.

Will the frog get across the river by jumping?

Laws/Results

Ⓜ A geometric series will converge to a limit (known as the *limiting sum*) iff

$$-1 < r < 1$$

7.2 Derivation of formula

Steps

1. Sum of a geometric series:

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

2. If $|r| < 1$, then $1 - r^n > 0$. Take limit as $n \to \infty$: (F)

7.3 Examples

Example 35

Explain why $2 + \sqrt{2} + 1 + \cdots$ has a limiting sum. Find the sum. **Answer:** $4 + 2\sqrt{2}$

Example 36

[2013 Independent 2U] Kevin has started an exercise program to lose weight. When he started the program he weighed 105 kg. In the first month he lost 5 kg, in the second month he lost 4 kg and in the third month he lost 3.2 kg. If this weight loss trend continues **Answer:** (i) 2.56 kg (ii) 80 kg

(i) how much will Kevin lose in the fourth month? **1**

(ii) what will be his ultimate weight? **3**

Example 37

[2010 NSB Prelim Ext 1 Yearly] **Answer:** (i) $-2 < x < -1$ (ii) $S = \frac{2x+3}{-2x-2}$

(i) (X1) For what values of x does this series have a limiting sum? **2**

$$2x + 3 + (2x+3)^2 + (2x+3)^3 + \cdots$$

(ii) Find that limiting sum in terms of x. **1**

Example 38

[2003 CSSA 2U] Consider the series $\sin^2 x + \sin^4 x + \sin^6 x + \cdots$, $0 < x < \frac{\pi}{2}$.

(i) Show that a limiting sum exists. **1**

(ii) Find the limiting sum, expressing your answer in simplest form. **2**

Example 39

[2011 Independent 2U] (3 marks) Find the value of x if

$$\sum_{n=0}^{\infty} \frac{9}{x^{n+1}} = 18$$

Answer: $x = \frac{3}{2}$

Example 40

[1998 2U] A ball is dropped from a height of 2 metres onto a hard floor and bounces. After each bounce, the maximum height reached by the ball is 75% of the previous maximum height. Thus, after it first hits the floor, it reaches a height of 1.5 metres before falling again, and after the second bounce, it reaches a height of 1.125 metres before falling again.

(i) What is the maximum height reached after the third bounce? **1**

(ii) What kind of sequence is formed by the successive maximum heights? **1**

(iii) What is the total distance travelled by the ball from the time it was first dropped until it eventually comes to rest on the floor? **2**

Example 41

[2003 2U]

(i) Find the limiting sum of the geometric series **2**

$$2 + \frac{2}{\sqrt{2}+1} + \frac{2}{\left(\sqrt{2}+1\right)^2} + \cdots$$

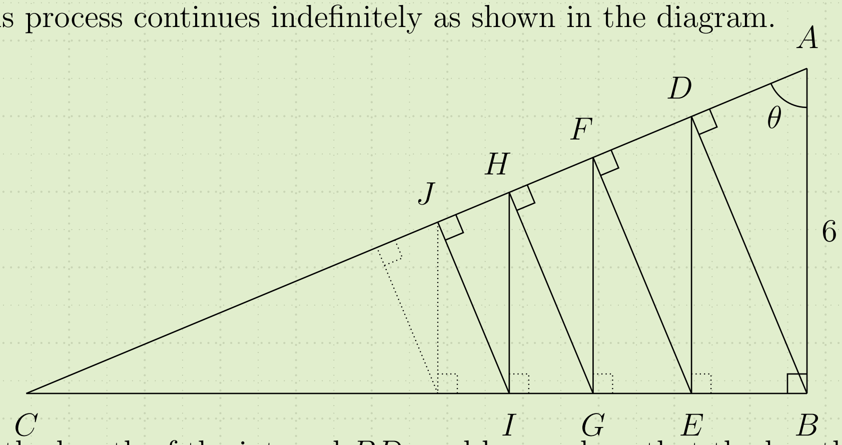
(ii) Explain why the geometric series **1**

$$2 + \frac{2}{\sqrt{2}-1} + \frac{2}{\left(\sqrt{2}-1\right)^2} + \cdots$$

does NOT have a limiting sum.

Example 42

[2005 2U Q9] The triangle ABC has a right angle at B, $\angle BAC = \theta$ and $AB = 6$. The line BD is drawn perpendicular to AC. The line DE is then drawn perpendicular to BC. This process continues indefinitely as shown in the diagram.



(i) Find the length of the interval BD, and hence show that the length of interval EF is $6\sin^3\theta$. **2**

(ii) Show that the limiting sum **3**

$$BD + EF + GH + \cdots$$

is given by $6\sec\theta\tan\theta$.

Example 43

[2014 2U HSC] At the beginning of every 8-hour period, a patient is given 10 mL of a particular drug.

During each of these 8-hour periods, the patient's body partially breaks down the drug. Only $\frac{1}{3}$ of the total amount of the drug present in the patient's body at the beginning of each 8-hour period remains at the end of that period.

(i) How much of the drug is in the patient's body immediately after the second dose is given? **1**

(ii) Show that the total amount of the drug in the patient's body never exceeds 15 mL. **2**

Example 44

[2017 2U HSC Q16] (3 marks) A geometric series has first term a and limiting sum 2.[a]

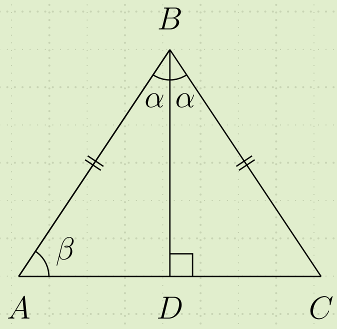
Find all possible values for a.

Answer: $0 < a < 4$

[a] Now an Extension 1 question

Example 45

[**1997 3U**] (X1) The triangle ABC is isosceles, with $AB = BC$, and BD is perpendicular to AC. Let $\angle ABD = \angle CBD = \alpha$ and $\angle BAD = \beta$, as shown in the diagram.



(i) Show that $\sin\beta = \cos\alpha$. **1**

(ii) By applying the sine rule in $\triangle ABC$, show that $\sin 2\alpha = 2\sin\alpha\cos\alpha$. **2**

(iii) Given that $0 < \alpha < \frac{\pi}{2}$, show that the limiting sum of the geometric series **2**

$$\sin 2\alpha + \sin 2\alpha\cos^2\alpha + \sin 2\alpha\cos^4\alpha + \sin 2\alpha\cos^6\alpha + \cdots$$

is equal to $2\cot\alpha$.

Further exercises

(2) **Ex 8H** (Pender et al., 2009)

- Q3-6, 12-16 last column
- Q7, 10, 11

(X1) **Ex 6K** (Pender et al., 1999)

- Q1-6 last column
- Q7 onwards

7.4 Application: recurring decimals

Example 46

Express the repeating decimals

1. $0.\overline{27}$ **2.** $2.6\overline{45}$

as infinite GPs, and use the formula for the limiting sum to find their values as fractions reduced to lowest terms. **Answer:** 1. $\frac{3}{11}$ 2. $\frac{291}{110}$

Important note

Formally in the legacy syllabus, and left here for application's sake.

Further exercises

(2) **Ex 8I** (Pender et al., 2009)

- all questions

(X1) **Ex 6L** (Pender et al., 1999)

- Q1-6 last 2 columns

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$$

$$S = \frac{a}{1 - r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

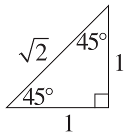
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

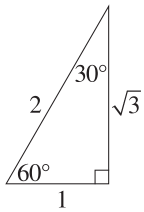
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

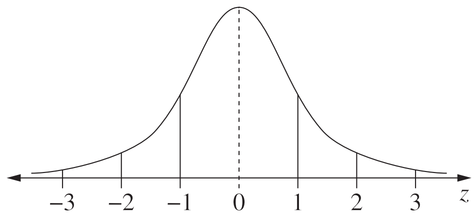
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^{n} = r^{n}(\cos n\theta + i\sin n\theta)$$
$$= r^{n}e^{in\theta}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^{2}(x-c)$$

References

Pender, W., Sadler, D., Shea, J., & Ward, D. (1999). *Cambridge Mathematics 3 Unit Year 11* (1st ed.). Cambridge University Press.

Pender, W., Sadler, D., Shea, J., & Ward, D. (2009). *Cambridge Mathematics 2 Unit Year 11* (2nd ed.). Cambridge University Press.